

Logical Relations and the Typed λ -Calculus

R. STATMAN

*Department of Mathematics, Carnegie-Mellon University,
Schenley Park, Pittsburgh, Pennsylvania*

INTRODUCTION

This paper consists of the previously unpublished material from a course of mine on the λ -calculus at Purdue in the spring of 1982.

The theme of the material is that the story of the (simple) typed λ -calculus is the story of a certain class of hereditarily defined relations here called “logical” relations. A notion of logical relation has already been introduced by Plotkin (1973) (see Plotkin, 1980). His notion is a special case of ours. The form of words ‘logical relation’ is particularly appropriate since the principal properties of such relations are closure under infinite conjunctions and existential quantifications, when these are suitably defined.

The notion of logical relation introduced below is a common generalization of Tait’s notion of a convertible term (Tait, 1967) (and its refinement by Prawitz, 1971; and Martin-Löf, 1973), and a whole host of hereditarily defined classes of functionals of which Howard’s hereditarily majorizable functionals [6] is typical.

Our intention is to first introduce the notion, and then proceed to prove, somewhat sketchily, the main syntactic results on the typed λ -calculus from the fundamental theorem of logical relations. These include the normalization and the Church–Rosser theorems. The Church–Rosser theorem is a case in point. Our proof, which is almost instantaneous, is the first we know of which makes essential use of the type structure. Next we obtain some of the principal semantic results on the typed λ -calculus as corollaries to the fundamental theorem. The first of these is a semantic version of this theorem. Others include the existence of free models and the completeness of \mathcal{P}_κ , the full type structure over a ground domain of size κ , when κ is infinite. These corollaries are not surprising since models are special sorts of logical relations (see Ex. 8 below).

Our main result is the characterization theorem. Both models of the typed λ -calculus and λ -definable elements of models are characterized in terms of logical relations. We also show that the characterization is in a

certain sense optimal. In particular, we show that Plotkin's Theorems 1 and 3 fail for general models.

We end the paper with some speculation on further possible uses of logical relations. Many of the results presented below are given as examples (of the uses of logical relations). This seems the appropriate format. It will be quite useful for the reader to be familiar with (Friedman, 1974; Plotkin, 1980; Statman, 1982) and the notation of (Statman, 1982). However, the main notational conventions are reviewed below.

The author would like to thank the referee for many useful corrections and comments.

NOTATION (in order of use)

(1) If Σ is a set of constants $A(\Sigma)$ is the set of all λ terms with constants from Σ and $A(\Sigma)^\tau$ the set of all $A(\Sigma)$ terms of type τ . $A = A(\phi)$.

(2) U, V, X, Y, Z are terms. M is a closed term in A . x, y, z are variables. $V \in \tau$ means: V has type τ .

(3) If τ is a type $\tau = \tau(1) \rightarrow (\cdots (\tau(k) \rightarrow \tau[k]) \cdots)$ for $1 \leq k \leq t$.

(4) λX is a λ -closure of X .

(5) $\|X\|_{\mathfrak{A}}(\alpha)$ = the value of X in \mathfrak{A} under the valuation α .
(assignment, context) α .

(6) $\beta\eta(X)$ = the $\beta\eta$ -normal form of X .

(7) All other notation is standard λ -calculus notation. (See Barendregt, 1981.)

LOGICAL RELATIONS

Let $\Sigma_1 \cdots \Sigma_n$ be sets of constants. $A(n)$ (n -ary) logical relation R is a map $\tau \rightarrow R_\tau \subseteq A(\Sigma_1)^\tau \times \cdots \times A(\Sigma_n)^\tau$ satisfying $R_\tau(X_1, \dots, X_n) \Leftrightarrow \forall Y_1 \cdots Y_n R_{\tau(1)}(Y_1, \dots, Y_n) \rightarrow R_{\tau[1]}(X_1 Y_1, \dots, X_n Y_n)$. Obviously, R is completely determined by R_0 .

EXAMPLE 1. Define R on A by $R_0 = \phi$. Then

$$\begin{aligned} R_\tau &= A^\tau && \text{if } \exists M \in \tau, \\ &= \phi && \text{otherwise.} \end{aligned}$$

The reader should compare this with (Lauchli, 1970).

EXAMPLE 2. Define R on $\mathcal{A}(\Sigma) \times \mathcal{A}(\Sigma)$ by $R_0(X, Y) \Leftrightarrow X =_{\beta\eta} Y$. Then $R(X, Y) \Leftrightarrow X =_{\beta\eta} Y$. R is said to be admissible if R_0 is closed under coordinatewise head expansions.

EXAMPLE 3. We say confluence holds from X if $Z \leq_{\beta\eta} X \geq_{\beta\eta} Y \Rightarrow \exists U Z \geq_{\beta\eta} U \leq_{\beta\eta} Y$. Define R by $R_0(X) \Leftrightarrow$ confluence holds from X . By the permutability of head contractions with internal contractions R is admissible. We shall continue this example. Our aim is to give a new proof of the Church–Rosser theorem.

Let $\theta_1 \cdots \theta_n$ be substitutions such that $\theta_i x \in \mathcal{A}(\Sigma_i)$ and let R be a logical relation on $\mathcal{A}(\Sigma_1) \times \cdots \times \mathcal{A}(\Sigma_n)$. We write $R(\theta_1, \dots, \theta_n)$ if $\forall x R(\theta_1 x, \dots, \theta_n x)$. Define R^* by $R^*(X_1, \dots, X_n) \Leftrightarrow \forall \theta_1 \cdots \theta_n, R(\theta_1, \dots, \theta_n) \rightarrow R(\theta_1 X_1, \dots, \theta_n X_n)$.

PROPOSITION 1. R^* is a logical relation which is admissible if R is admissible.

Proof. The second part is obvious. For the first part, first assume $R_\tau^*(X_1, \dots, X_n)$ and $R_{\tau(1)}^*(Y_1, \dots, Y_n)$. Let $\theta_1 \cdots \theta_n$ satisfy $R(\theta_1, \dots, \theta_n)$, then $R_{\tau(1)}(\theta_1 Y_1, \dots, \theta_n Y_n)$ and $R_\tau(\theta_1 X_1, \dots, \theta_n X_n)$. Thus $R_{\tau[1]}(\theta_1 X_1 Y_1, \dots, \theta_n X_n Y_n)$, so $R_{\tau[1]}^*(X_1 Y_1, \dots, X_n Y_n)$. Conversely, suppose $\forall Y_1 \cdots Y_n R_{\tau(1)}^*(Y_1, \dots, Y_n) \rightarrow R_{\tau[1]}^*(X_1 Y_1, \dots, X_n Y_n)$. Let $\theta_1 \cdots \theta_n$ satisfy $R(\theta_1, \dots, \theta_n)$ and select $x \in \tau(1)$ not among the free variables of X_1, \dots, X_n . Since $R_{\tau(1)}^*(x, \dots, x)$, $R_{\tau[1]}^*(X_1 x, \dots, X_n x)$. Suppose $R_{\tau(1)}(Y_1, \dots, Y_n)$. Define

$$\begin{aligned} \theta_i^* y &\equiv \theta_i y & y &\neq x, \\ &\equiv Y_i & y &\equiv x. \end{aligned}$$

Then $R(\theta_1^*, \dots, \theta_n^*)$ so $R_{\tau[1]}(\theta_1^* X_1 x, \dots, \theta_n^* X_n x)$. That is, $R_{\tau[1]}((\theta_1 X_1) Y_1, \dots, (\theta_n X_n) Y_n)$. Hence $R_\tau(\theta_1 X_1, \dots, \theta_n X_n)$, so $R_\tau^*(X_1, \dots, X_n)$.

R is called regular if it is admissible and is closed under coordinatewise head contractions at type 0. If R is regular then $R^*(X_1, \dots, X_n) \Leftrightarrow R(\lambda X_1, \dots, \lambda X_n)$.

If C is a class of logical relations over $\mathcal{A}(\Sigma_1) \times \cdots \times \mathcal{A}(\Sigma_n)$, define $AC(X_1, \dots, X_n) \Leftrightarrow \forall R \in C R^*(X_1, \dots, X_n)$.

PROPOSITION 2. AC is a logical relation which is admissible if each $R \in C$ is admissible.

Proof. The second part is obvious. In addition, it is obvious that $AC_\tau(X_1, \dots, X_n) \wedge AC_{\tau(1)}(Y_1, \dots, Y_n) \rightarrow AC_{\tau[1]}(X_1 Y_1, \dots, X_n Y_n)$. Suppose that $\forall Y_1 \cdots Y_n AC_{\tau(1)}(Y_1, \dots, Y_n) \rightarrow AC_{\tau[1]}(X_1 Y_1, \dots, X_n Y_n)$. Select $x \in \tau(1)$ but not among the free variables of X_1, \dots, X_n . Then $AC_{\tau(1)}(x, \dots, x)$ so

$AC_{\tau[1]}(X_1 x, \dots, X_n x)$. Fix $R \in C$ and suppose $R_{\tau(1)}(Y_1, \dots, Y_n)$. Let $\theta_1 \cdots \theta_n$ satisfy $R(\theta_1, \dots, \theta_n)$. Then $R_{\tau(1)}(\theta_1 Y_1, \dots, \theta_n Y_n)$. Define θ_i by

$$\begin{aligned} \theta_i y &\equiv \theta_i y, & y &\neq x, \\ &\equiv \theta_i Y_i, & y &\equiv x. \end{aligned}$$

We have $R(\theta_1, \dots, \theta_n)$ so $R_{\tau[1]}(\theta_1 X_1 x, \dots, \theta_n X_n x)$. That is $R_{\tau[1]}(\theta_1 X_1 Y_1, \dots, \theta_n X_n Y_n)$. Thus $R_{\tau}^*(X_1, \dots, X_n)$ so $AC_{\tau}(X_1, \dots, X_n)$.

If R is a logical relation over $A(\Sigma_1) x \cdots x A(\Sigma_n)$ define $\exists R$ by $\exists R(X_1, \dots, X_{n-1}) \Leftrightarrow \exists X_n R^*(X_1, \dots, X_{n-1}, X_n)$.

PROPOSITION 3. If R is admissible then $\exists R$ is an admissible logical relation.

Proof. As before we need only verify that if $\forall Y_1 \cdots Y_{n-1}$, $\exists R_{\tau(1)}(Y_1, \dots, Y_{n-1}) \rightarrow \exists R_{\tau[1]}(X_1 Y_1, \dots, X_{n-1} Y_{n-1})$ then $\exists R_{\tau}(X_1, \dots, X_{n-1})$. Select $x \in \tau(1)$ but not among the free variables of $X_1 \cdots X_{n-1}$. Since $R_{\tau(1)}^*(x, \dots, x)$, $\exists X_n R_{\tau[1]}^*(X_1 x, \dots, X_{n-1} x, X_n)$. Since R is admissible R^* is admissible, so $R_{\tau[1]}^*(X_1 x, \dots, X_{n-1} x, (\lambda x X_n) x)$. Now suppose $R_{\tau(1)}^*(Y_1, \dots, Y_n)$. Let $\theta_1 \cdots \theta_n$ satisfy $R(\theta_1, \dots, \theta_n)$. Then $R_{\tau(1)}(\theta_1 Y_1, \dots, \theta_n Y_n)$. Define θ_i^+ by

$$\begin{aligned} \theta_i^+ y &\equiv \theta_i y & \text{if } y &\neq x, \\ &\equiv \theta_i Y_i & \text{if } y &\equiv x. \end{aligned}$$

Then $R(\theta_1^+, \dots, \theta_n^+)$ so $R_{\tau[1]}(\theta_1^+ X x, \dots, \theta_{n-1}^+ X_{n-1} x, \theta_n^+ (\lambda x X_n) x)$. That is, $R_{\tau[1]}(\theta_1 X_1 Y_1, \dots, \theta_{n-1} X_{n-1} Y_{n-1}, \theta_n \lambda x X_n Y_n)$. Hence $R_{\tau}^*(X_1, \dots, X_{n-1}, \lambda x X_n)$, so $\exists R_{\tau}(X_1, \dots, X_{n-1})$.

PROPOSITION 4. (1) $R^{**} = R^*$

(2) $AC^* = AC$

(3) If $\exists R$ is logical then $(\exists R)^* = \exists R$.

Proof. Similar to the proofs of Propositions 1–3. Define R^W on $A(\Sigma_1) \times \cdots \times A(\Sigma_n)$ by $R^W(X) \Leftrightarrow R^*(X, \dots, X)$. R^W is logical, and admissible if R is admissible.

FUNDAMENTAL THEOREM OF LOGICAL RELATIONS. If R is admissible then

$$X \in A \Rightarrow R^W(X).$$

Proof. The proof is by induction on X . We give only the induction step

when $X \equiv \lambda y Y$. Suppose $R_{\tau(1)}^*(Z_1, \dots, Z_n)$ and $R(\theta_1, \dots, \theta_n)$. Then $R_{\tau(1)}(\theta_1 Z_1, \dots, \theta_n Z_n)$. Define $\# \theta_i$ by

$$\begin{aligned} \theta_i^\# x &\equiv \theta_i x && \text{if } x \neq y, \\ &\equiv \theta_i Z_i && \text{if } x \equiv y. \end{aligned}$$

Then $R(\theta_1^\#, \dots, \theta_n^\#)$, the induction hypothesis is $R_{\tau[1]}^*(Y, \dots, Y)$, so $R_{\tau[1]}(\theta_1^\# Y, \dots, \theta_n^\# Y)$. Since R is admissible $R_{\tau[1]}((\theta_1 X)(\theta_1 Z_1), \dots, (\theta_n X)(\theta_n Z_n))$, i.e.,

$$R_{\tau[1]}(\theta_1 X Z_1, \dots, \theta_n X Z_n). \text{ Thus } R_{\tau[1]}^*(X Z_1, \dots, X Z_n) \text{ and } R^w(X).$$

EXAMPLE 3 (continued). Let R be as in Example 3. We prove simultaneously by induction on τ that $x \in \tau \rightarrow R_\tau(x)$ and $R_\tau(X) \rightarrow$ confluence holds from X . Only the second part is nontrivial. Suppose $R_\tau(X)$ and $Y <_{\beta\eta} X \geq_{\beta\eta} Z$. By induction hypothesis, for $i = 1, \dots, t$ and $x_i \in \tau(i)$, $R_{\tau(i)}(x_i)$, and thus confluence holds from $X x_1 \cdots x_t$. We have $Y x_1 \cdots x_t \leq_{\beta\eta} X x_1 \cdots x_t \geq_{\beta\eta} Z x_1 \cdots x_t$ so there exists U such that $Y x_1 \cdots x_t \geq_{\beta\eta} U \leq_{\beta\eta} Z x_1 \cdots x_t$. By skipping head contractions with arguments x_i there exist $\lambda y_1 \cdots y_k V_1$, $\lambda z_1 \cdots z_m V_2$ such that $Y \geq_{\beta\eta} \lambda y_1 \cdots y_k V_1$, $Z \geq_{\beta\eta} \lambda z_1 \cdots z_m V_2$, and $[x_1/y_1, \dots, x_k/y_k] V_1 x_{k+1} \cdots x_t \equiv U \equiv [x_1/z_1, \dots, x_m/z_m] V_2 x_{m+1} \cdots x_t$. W.l.o.g. $k \leq m$ so $\lambda y_1 \cdots y_k V_1 \leq_\eta \lambda z_1 \cdots z_m V_2$. Thus confluence holds from X . Thus by the fundamental theorem of logical relations $X \in \mathcal{A} \Rightarrow$ confluence holds from X .

EXAMPLE 4. Tait's proof (Tait, 1967) of the normalizability of λK terms as modified by Prawitz (1971) can be literally copied into this context using the fundamental theorem. In addition, a proof similar to the one in Example 3 can be given for the strong normalizability of λI terms. The strong normalizability of λK terms follows from this by a trick of Gandy (1980). The difficulty in a direct proof is showing that R , where R_0 = the set of strongly normalizable terms of type 0, is admissible. This, for the λI -calculus, is just as in example 3. The reader should compare this argument with (Gandy, 1980) (perhaps after reading Example 8 below) and again with (Prawitz, 1971). Finally, proofs similar to the one in Example 3 can be given for termination of standard reductions, completeness of standard reductions, and η postponement.

EXAMPLE 5. Consider the λI -calculus and let R_0 be the generalized Higman ordering of (Statman, 1900) on $\mathcal{A}^0 \times \mathcal{A}^0$. Let \bar{R}_τ be the restriction of R_τ to closed terms. It is shown in (Statman, 1900, 13, Lemma 16] that $\bar{R}_{((0 \rightarrow 0) \rightarrow 0) \rightarrow (0 \rightarrow 0)}$ is not a well partial ordering. A set S of closed terms is finitely generated if there exists a finite set F of closed terms such that

every member of $S \beta \eta$ converts to an applicative combination of members of F . One can prove easily by the methods of (Higman, 1952) that if S is a set of closed terms of the same type which is finitely generated then S is well partially ordered by \bar{R} . Thus the set of closed terms of type $((0 \rightarrow 0) \rightarrow 0) \rightarrow 0$ is not finitely generated. The corresponding problem for λK is open.

LOGICAL RELATIONS ON TYPE STRUCTURES

A frame \mathfrak{A} (called “prestructure” in [2]) is a map $\tau \rightarrow \mathfrak{A}_\tau$ from types to nonempty sets satisfying $\mathfrak{A}_\tau \subseteq \mathfrak{A}_{\tau[1]}^{\mathfrak{A}_{(1)}}$. If \mathfrak{A} is a frame then \mathfrak{A}^* is obtained from \mathfrak{A} by adjoining infinitely many indeterminates (variables) of each type to \mathfrak{A} . An \mathfrak{A} valuation is a retract of \mathfrak{A}^* onto \mathfrak{A} , i.e., a (total) homomorphism of \mathfrak{A}^* onto \mathfrak{A} which fixes each element of \mathfrak{A} . In particular, for each $\Phi, \Psi \in \mathfrak{A}^*$ $\Phi = \Psi \Leftrightarrow$ for all \mathfrak{A} valuations α , $\alpha(\Phi) = \alpha(\Psi)$.

\mathfrak{A} is a model (of the typed λ -calculus) if $\forall X \forall \alpha \parallel X \parallel_{\mathfrak{A}}(\alpha)$ exists. (See Friedman, 1974.) The following is an easy exercise. If \mathfrak{A} is a frame, $\parallel X \parallel_{\mathfrak{A}}(\alpha)$ exists, and $X \beta \eta$ red. Y then $\parallel Y \parallel_{\mathfrak{A}}(\alpha)$ exists and is $\parallel X \parallel_{\mathfrak{A}}(\alpha)$.

If $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ are frames $a(n)$ (n -ary) logical relation R is a map $\tau \rightarrow R_\tau \subseteq \mathfrak{A}_1^\tau \times \dots \times \mathfrak{A}_n^\tau$ satisfying $R_\tau(\Phi_1, \dots, \Phi_n) \Leftrightarrow \forall \Psi_1 \dots \Psi_n R_{\tau(1)}(\Psi_1, \dots, \Psi_n) \rightarrow R_{\tau[1]}(\Phi_1 \Psi_1, \dots, \Phi_n \Psi_n)$.

EXAMPLE 6. The equality relation of \mathfrak{A} is a logical relation on $\mathfrak{A} \times \mathfrak{A}$.

EXAMPLE 7. If R is the graph of a partial surjective homomorphism from \mathfrak{A} onto \mathcal{L} (see Friedman, 1974), then R is a logical relation on $\mathfrak{A} \times \mathcal{L}$. Conversely, if R is a logical relation on $\mathfrak{A} \times \mathcal{L}$ such that R_0 is the graph of a partial surjective map from \mathfrak{A}_0 onto \mathcal{L}_0 and $\forall \Psi \exists \Phi R(\Phi, \Psi)$, then R is the graph of a partial homomorphism from \mathfrak{A} onto \mathcal{L} .

If α_i is an \mathfrak{A}_i valuation we write $R(\alpha_1, \dots, \alpha_n)$; if for each indeterminate x , $R(\alpha_1(x), \dots, \alpha_n(x))$. Define R^* by $R^*(\Phi_1, \dots, \Phi_n) \Leftrightarrow \forall \alpha_1 \dots \alpha_n, R(\alpha_1, \dots, \alpha_n) \rightarrow R(\alpha_1(\Phi), \dots, \alpha_n(\Phi_n))$. Similarly, AC (for C a class of logical relations) $\exists R$ and R^w are defined as before.

PROPOSITION 5. (1) R^* is a logical relation on $\mathfrak{A}_1^* \times \dots \times \mathfrak{A}_n^*$ and $R^{**} = R^*$.

(2) AC is a logical relation on $\mathfrak{A}_1^* \times \dots \times \mathfrak{A}_n^*$ and $AC^* = AC$.

(3) If \mathfrak{A}_n is a model then $\exists R$ is a logical relation on $\mathfrak{A}_1^* \times \dots \times \mathfrak{A}_{n-1}^*$. If $\exists R$ is a logical relation then $(\exists R)^* = \exists R$.

EXAMPLE 8. Many hereditarily defined classes of functionals in the literature are logical relations such as the hereditarily majorizable functionals (Howard, 1973) and the invariant functionals (Lauchli, 1970). The hereditarily monotonic functionals (Gandy, 1980) [3]) are not.

The $*$ operation cannot be eliminated from the definition of \mathcal{AC} and $\exists R$. For \mathcal{AC} this is an exercise. For $\exists R$, suppose that $\mathfrak{U}_0 = N$ and $\mathfrak{U}_{0 \rightarrow 0}$ contains s but only primitive recursive functions. Let R_0 be the graph of a non-primitive recursive function. Since $\forall n \exists m R_0(n, m)$ we have $\exists m R_0(n, m) \rightarrow \exists m R_0(sn, m)$. However, $\neg \exists \Phi R_{0 \rightarrow 0}(s, \Phi)$.

If E is the equality relation on \mathfrak{U} then E^* is the equality relation on \mathfrak{U}^* . If H is the graph of a partial surjective homomorphism of \mathfrak{U} onto \mathcal{L} then H^* is the graph of the partial surjective homomorphism from \mathfrak{U}^* onto \mathcal{L}^* whose restriction to \mathfrak{U} is H and which fixes each indeterminate.

Let $\Sigma = \{A_\Phi : \Phi \in \mathfrak{U}\}$, where $\|A_\Phi\|_{\mathfrak{U}}(\alpha) = \Phi$. For what follows it is convenient to use $\mathcal{A}(\Sigma)$ rather than the corresponding free model of $\beta\eta$ conversion. Define D on $\mathcal{A}(\Sigma) \times \mathfrak{U}$ by $D(X, \Phi) \Leftrightarrow \forall \alpha \| \beta\eta(X) \|_{\mathfrak{U}}(\alpha)$ exists and is Φ .

PROPOSITION 6. D is a logical relation.

Proof. First suppose that $D_{\tau}(X, \Phi)$ and $D_{\tau(1)}(Y, \Psi)$. Then we have $\| \beta\eta(X) \beta\eta(Y) \|(\alpha)$ exists and $= \| \beta\eta(X) \|(\alpha) \| \beta\eta(Y) \|(\alpha) = \Phi\Psi$. Thus, since $\beta\eta(X) \beta\eta(Y) \geq_{\beta\eta} \beta\eta(XY) \| \beta\eta(XY) \|(\alpha)$ exists and is $\Phi\Psi$. That is, $D_{\tau[1]}(XY, \Phi\Psi)$. Conversely, suppose that $\forall Y\Psi$, $D_{\tau(1)}(Y, \Psi) \rightarrow D_{\tau[1]}(XY, \Phi\Psi)$. Then for each $\Psi \in \mathfrak{U}_{\tau(1)} \forall \alpha$, $\| \beta\eta(XA_\Psi) \|(\alpha)$ exists and is $\Phi\Psi$.

Case 1. $\beta\eta(X)$ does not begin with λ . Then $\beta\eta(XA_\Psi) = \beta\eta(X) A_\Psi$ so $\| \beta\eta(X) \|(\alpha)$ exists and $\| \beta\eta(X) \|(\alpha) \Psi = \Phi\Psi$. Thus $\| \beta\eta(X) \|(\alpha) = \Phi$.

Case 2. $\beta\eta(X) = \lambda y Y$. Then $\beta\eta(XA_\Psi) = [A_\Psi/y] Y$. Thus for each α , $\| [A_\Psi/y] Y \|(\alpha)$ exists and is $\Phi\Psi$. But $\| [A_\Psi/y] Y \|(\alpha) = \| Y \|([\Psi/y] \alpha)$, where

$$\begin{aligned} [\Psi/y] \alpha(x) &= \alpha(x) & \text{if } x \neq y, \\ &= \Psi & \text{if } x \equiv y. \end{aligned}$$

Thus $\| \beta\eta(X) \|(\alpha)$ is Φ .

If R is a logical relation on $\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$ define \hat{R} on $\mathcal{A}(\Sigma_1) \times \cdots \times \mathcal{A}(\Sigma_n)$ by

$$\hat{R}(X_1, \dots, X_n) \Leftrightarrow \exists \Phi_1 \cdots \Phi_n D^*(X_1, \Phi_1) \wedge \cdots \wedge D^*(X_n, \Phi_n) \wedge R^*(\Phi_1, \dots, \Phi_n).$$

If R is a logical relation on $\mathcal{A}(\Sigma_1) \times \cdots \times \mathcal{A}(\Sigma_n)$ define \check{R} on $\mathfrak{U}_1^* \times \cdots \times \mathfrak{U}_n^*$ by

$$\check{R}(\Phi_1, \dots, \Phi_n) \Leftrightarrow \exists X_1 \cdots X_n D^*(X_1, \Phi_1) \wedge \cdots \wedge D^*(X_n, \Phi_n) \wedge R^*(X_1, \dots, X_n).$$

PROPOSITION 7. (1) If $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ are models then \hat{R} is an admissible logical relation and $\hat{R} = R^*$.

(2) If R is an admissible logical relation then \check{R} is a logical relation. Moreover, if $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ are models then $\check{R} = R^*$.

FUNDAMENTAL THEOREM OF LOGICAL RELATIONS (Version 2). If $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ are models of the typed λ -calculus and R is a logical relation on $\mathfrak{A}_1 \times \dots \times \mathfrak{A}_n$ then for each M

$$R^*(\|M\|_{\mathfrak{A}_1, \dots, \mathfrak{A}_n}).$$

EXAMPLE 7 (continued). Let H be the graph of a partial homomorphism from \mathfrak{A} onto \mathcal{L} . Applying the fundamental theorem, if \mathfrak{A} and \mathcal{L} are models, gives Lemma 15 of Friedman (1974). Thus we obtain Friedman's completeness theorem (Friedman, 1974, Theorem 3). More generally, we obtain Proposition 1 of (Plotkin, 1980).

EXAMPLE 8. Suppose \mathfrak{A} is a model and E is the equality relation on \mathfrak{A} . Put $[X] = \{Y: \hat{E}(X, Y)\}$ and $[X][Y] = [XY]$. Then the resulting frame is isomorphic to \mathfrak{A}^* . Conversely, suppose that R is an admissible logical relation on $A(\Sigma) \times A(\Sigma)$ such that R_0 is a nonempty equivalence relation on its field. Then, putting $[X] = \{Y: R(X, Y)\}$ and $[X][Y] = [XY]$, the resulting frame is a model. Now suppose that S is a set of equations. Define R by $R_0(X, Y) \Leftrightarrow S \vdash X = Y$ where \vdash refers to the rules 1–5 of (Friedman, 1974). (Here rule 5 should be modified by “ x not free in any assumption.”)

By a simple induction on types $R_i(X, Y) \Leftrightarrow S \vdash X = Y$. Thus we obtain the completeness theorem (Theorem 1) of Statman (1982).

EXAMPLE 9. Let \mathfrak{A} be a model and \leq a partial ordering of \mathfrak{A}_0 . Define $R_0^a = \{a\}$ for $a \in \mathfrak{A}_0$ and let $C = \{R^a: a \in \mathfrak{A}_0\}$. Put $I(\Phi) \Leftrightarrow \exists \Psi \Phi^* \leq \Psi \wedge AC(\Psi)$. Then for $\Phi \in \mathfrak{A}_{0 \rightarrow 0}$, $I(\Phi) \Leftrightarrow \Phi$ is increasing (i.e., $a \leq \Phi a$).

CHARACTERIZATION THEOREM. (1) \mathfrak{A} is a model if and only if for each \mathcal{L} and logical relation R on $\mathcal{L} \times \mathfrak{A}$ $\exists R$ is logical.

(2) If \mathfrak{A} is a model and $\Phi \in \mathfrak{A}$ then Φ is λ -definable if and only if for each logical relation R on \mathfrak{A}^* $R(\Phi)$.

Proof of (1). \Rightarrow is Proposition 5(3). For \Leftarrow take R to be D . Then $\exists \Phi D^*(X, \Phi)$ is a logical relation on $A(\Sigma)$. Thus, by Proposition 5(3) and the fundamental theorem, for each $M \in A$, $\exists \Phi D^*(M, \Phi)$. Thus for each $M \in A$, $\exists \Phi \|M\|$ exists and is Φ .

Proof of (2). See Statman (1900).

The reader should compare (2) with Theorem 1 of Plotkin (1973). We shall now show that the $*$ cannot be removed from (2). $\Phi : N \rightarrow N$ is called regressive with turning point n if for each $m \geq n$ $\Phi : [0, m] \rightarrow [0, m]$. That is, $\forall k \leq n \Phi(k) \leq n \wedge \forall k > n \Phi(k) \leq k$. Let U be a nonprincipal ultrafilter on N . Let \lim be any member of $(N \rightarrow N) \rightarrow (N \rightarrow N)$ such that for regressive Φ $(\lim \Phi) n = m \Leftrightarrow \{k : \Phi^k n = m\} \in U$. Let P be the full type structure with ground domain N . Define logical relations R_n, S_n by $R_{n,0} = [0, n]$ and $S_n = \bigcap_{n \leq m} R_m^*$:

(1) For $k \in N$, $S_{n,0}(k) \Leftrightarrow k \leq n$.

(2) For $\Phi \in P_{0 \rightarrow 0}$, $S_n(\Phi) \Leftrightarrow \Phi$ is regressive with turning point n .

(3) $S_n(\lim)$, for suppose $S_{n,0 \rightarrow 0}(\Phi)$, by Proposition 5(2), $S_n^* = S_n$. Suppose $S_n(\alpha)$, then $S_n(\alpha(\Phi))$, so $\alpha(\Phi)$ is regressive with turning point n . Thus $\lim \alpha(\Phi)$ is regressive with turning point n . Hence $S_n(\alpha(\lim \Phi))$. Thus $S_n(\lim)$.

(4) If $\Phi \in P_{0 \rightarrow 0}$ is λ -definable from \lim , regressive functions and members of N then Φ is regressive.

For, if $\Phi = \|M\| \Phi_1 \cdots \Phi_n \lim 0 \cdots m$ let Φ_i have turning point m_i . Put $k = \max\{m_i, 1 \leq i \leq n\}$. By (1), (2), (3) and the fundamental theorem $S_k(\Phi)$. Thus Φ is regressive by (2). Let \mathfrak{A} = the Gandy hull (see Statman, 1982) of N , \lim , and all regressive functions in P .

(5) Let R be an n -ary I-logical relation on $\mathfrak{A} \times \cdots \times \mathfrak{A} \times w$ (see Plotkin, 1980). Suppose $R_{0 \rightarrow 0}(\Phi_1, \dots, \Phi_n, w)$, $w \leq w'$, and $R_0(m_1, \dots, m_n, w')$. Let $T_i = \{k : (\lim \Phi_i) m_i = \Phi_i^k m_i\}$. Since each $T_i \in U$, $\bigcap_{1 \leq i \leq n} T_i \in U$. Let $k \in \bigcap_{1 \leq i \leq n} T_i$. We have $R_0(\Phi_1^k m_1, \dots, \Phi_n^k m_n, w')$ thus $R_0((\lim \Phi_1) m_1, \dots, (\lim \Phi_n) m_n, w')$. Hence $R_{0 \rightarrow 0}(\lim \Phi_1, \dots, \lim \Phi_n, w)$. Thus $R(\lim, \dots, \lim, w)$ for all w . In other words, \lim satisfies every I-logical relation.

(6) Suppose $\lim = \|M\|$ for M in long $\beta\eta$ normal form. Then $M \equiv \lambda xy x(\underbrace{\cdots (xy) \cdots}_n)$ for some n . Let $\Phi = \lambda x x - 1$. Φ is regressive but

$(\lim \Phi) n + 1 = 1$. This contradicts the choice of U , since $\{k : \Phi^k n + 1 = 0\}$ is cofinite. In other words, \lim is not λ -definable.

In particular, we have shown that Plotkin's (1980) Theorems 1 and 3 fail for general models. However, no example of this sort is possible for hereditary finite models.

For suppose \mathfrak{A} is a model and $|\mathfrak{A}_0|$ is finite. Suppose we are given $\lim \varepsilon \mathfrak{A}(0 \rightarrow 0) \rightarrow (0 \rightarrow 0)$. Let $U_a^\Phi = \{k : \Phi^k a = \lim \Phi a\}$ for each $\Phi \in \mathfrak{A}_{0 \rightarrow 0}$ and $a \in \mathfrak{A}_0$. Suppose $U_a^\Phi \cap U_b^\Psi \neq \emptyset$ for all Φ, Ψ, a, b . Then $\bigcap_{\Phi, a} U_a^\Phi \neq \emptyset$. For if $|U_a^\Phi| > 1$ then U_a^Φ is ultimately periodic, i.e., U_a^Φ is an arithmetic progression modulo a finite set. Thus either $\bigcap_{\Phi, a} U_a^\Phi$ is a singleton, or by the Chinese remainder theorem $\bigcap_{\Phi, a} U_a^\Phi$ is an arithmetic progression

modulo a finite set. In particular, for $k \in \bigcap_{\phi, a} U_a^\phi$, $\lim = \|\lambda xy x(\underbrace{\cdots (xy) \cdots})_k\|$.

Plotkin's Theorem 1 does fail for hereditarily finite full models but Theorem 3 remains open. For more about this case we refer the reader to Statman (with Saks, 1900).

EXAMPLE 10. There are I -logical relations which are not logical relations. For example, let $W = N$ and define R on $P \times W$ by $R_0(n, m) \Leftrightarrow n \leq m$. Let $S(\Phi) \Leftrightarrow S(\Phi, 0)$ so $S_0 = \{0\}$. Thus $S_{0 \rightarrow 0}(\lambda x 2x)$ if S is logical, and $2 \leq 1$. However, I -logical relation can be reduced to logical ones as follows.

We illustrate with unary I -logical relations R on $\mathfrak{U} \times W$. First let W^+ be W with a top element T and a bottom element \perp . Define

$$\begin{aligned} R^+(\Phi, u) &\Leftrightarrow u = \perp \wedge \forall w \in WR(\Phi, w) \vee \\ &u \in W \wedge R(\Phi, u) \vee \\ &u = T. \end{aligned}$$

Then R^+ is an I -logical relation on $\mathfrak{U} \times W^+$. Define a frame \mathcal{L} recursively as follows. $[w]_0 = w$ for $w \in W^+$ and $\mathcal{L}_0 = \{[w]_0 : w \in W^+\}$. If $w_1 \geq w_2$ then $[w_2]_\tau [w_1]_{\tau(1)} = [w_1]_{\tau[1]}$ else $[w_2]_\tau [w_1]_{\tau(1)} = [T]_{\tau[1]}$ and $\mathcal{L}_\tau = \{[w]_\tau : w \in W^+\}$. Define $R^\#$ on $\mathfrak{U} \times \mathcal{L}$ by $R^\#(\Phi, [w]) \Leftrightarrow R^+(\Phi, w)$. Then $R^\#$ is a logical relation on $\mathfrak{U} \times \mathcal{L}$.

EXAMPLE 11. Every frame can be freely extended to a model as follows. Given \mathfrak{U} put $\Sigma = \{A_\phi : \phi \in \mathfrak{U}\}$. Extend the rules for $\beta\eta$ reducibility by adding the rule $(\mathfrak{U}) A_\phi A_\psi$ red, $A_{\phi\psi}$. Then $\beta\eta\mathfrak{U}$ normal forms exist and are unique (Hindley, 1969). We now show how to extend logical relations. We illustrate with a unary logical relation on \mathfrak{U} .

Let R be logical on \mathfrak{U} and \mathfrak{U}^λ the extended model. For what follows it is convenient to use $\lambda(\Sigma)$ rather than the corresponding free model of $\beta\eta\mathfrak{U}$ conversion. Define R^λ on $\lambda(\Sigma)$ by

$$R_0^\lambda(X) \Leftrightarrow \forall a \in \mathfrak{U}_c(\beta\eta\mathfrak{U}(X)) = A_a \rightarrow R_0(a).$$

One verifies easily that if $\beta\eta\mathfrak{U}(X) = A_\phi$ then $R^\lambda(X) \Leftrightarrow R(\phi)$.

LATTICES OF LOGICAL RELATIONS

Remark. If $\mathfrak{U}_1 \cdots \mathfrak{U}_n$ are models then $(\mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n)^*$ is isomorphic to $\mathfrak{U}_1^* \times \cdots \times \mathfrak{U}_n^*$. Actually, the existence of projections suffices. For the sake of brevity it is convenient to identify the two.

If \mathfrak{A} is a model and C a collection of logical relations on \mathfrak{A} define $VC = \text{applicative closure of } \bigcup_{R \in C} R^*$.

PROPOSITION 8. VC is logical and $VC^* = VC$.

Proof. Obviously $VC_\tau(\Phi) \wedge VC_{\tau(1)}(\Psi) \Rightarrow VC_{\tau[1]}(\Phi\Psi)$. Suppose $\forall \Psi$, $VC_{\tau(1)}(\Psi) \rightarrow VC_{\tau[1]}(\Phi\Psi)$. Pick $x \in \tau(1)$ but not among the free variables of Φ . Since $R^*(x)$ for each $R \in C$, we have $VC_{\tau[1]}(\Phi x)$. Thus there exists $\Phi_1 \cdots \Phi_m \in \mathfrak{A}^*$ s.t. for each Φ_i there exists $R_i \in C$ s.t. $R_i^*(\Phi_i)$ and Φx is an applicative combination of the Φ_i . Hence there exists a closed term M s.t. $\Phi = \|M\|_{\mathfrak{A}} \lambda x \Phi_1 \cdots \lambda x \Phi_m$. By the fundamental theorem $\forall R \in C$, $R^*(\|M\|_{\mathfrak{A}})$. Moreover, $R_i^*(\lambda x \Phi_i)$. Thus $VC_\tau(\Phi)$.

For the second part. Suppose $\forall \alpha$, $VC(\alpha) \rightarrow VC(\alpha(\Phi))$. Putting $\alpha = id$ we get $VC(\Phi)$. Conversely, suppose $VC(\Phi)$. Let $VC(\alpha)$. Then $VC(\alpha(\Phi))$.

Let L_n be the set of n -ary logical relations R on \mathfrak{A}^* satisfying $R^* = R$. For $F \subseteq \mathfrak{A}^{n*}$ let $A^*(F)$ be the set of all $\Phi \in \mathfrak{A}^{n*}$ λ definable from the λ closures of members of F and indeterminates. We have

PROPOSITION 9. L_n is a complete algebraic lattice whose compact elements have the form $A^*(F)$ for finite F .

INNER MODELS (Speculation)

Suppose R is a logical relation on $\mathcal{L} \times \mathcal{L}$ such that R is an equivalence relation on its field $F(R_0 \neq \emptyset)$. Define $[\Phi] = \{\Psi : R(\Phi, \Psi)\}$ and $[\Phi][\Psi] = [\Phi\Psi]$ for $\Phi, \Psi \in F$. Then $\mathcal{L}_R = \{[\Phi] : \Phi \in F\}$ is a frame (as in Ex. 8). \mathfrak{A} is called an inner frame of \mathcal{L} if \mathfrak{A} is isomorphic to \mathcal{L}_R for some such R .

EXAMPLE 12. \mathfrak{A} is an inner model of \mathfrak{A}^* , where $R_0(a, b) \Leftrightarrow a = b \in \mathfrak{A}_0$.

PROPOSITION 10. If there is a partial surjective homomorphism from \mathfrak{A} onto \mathcal{L} and \mathcal{L} is a model then \mathcal{L}^* is inner model of \mathfrak{A}^* .

Proof. Suppose H is such a partial homomorphism. As in Example 7 define $R(\Phi_1, \Phi_2, \Psi) \Leftrightarrow H(\Phi_1) = \Psi = H(\Phi_2)$ so R is logical. Then \mathcal{L}^* is isomorphic to $\mathfrak{A}_{\exists R}^*$.

In particular, if P_κ is the full type structure over a ground domain of size κ by (Friedman, 1974, Lemma 17) if \mathfrak{A} is a model and $|\mathfrak{A}_0| \leq \kappa$ then \mathfrak{A}^* is an inner model of P_κ^* .

EXAMPLE 13. $PP\lambda$ (Barendregt, 1981) and A.C. (Andrews, 1972) are mutually inconsistent. In particular, for each theory there is a functional equation solvable in every model of the theory and in no model of the other theory. Namely, $PP\lambda \vdash \exists \Phi, \lambda x x(\Phi x) = \Phi$ and $A.C. \vdash \exists \Phi \Psi \lambda x \Phi x(\Psi(x\Psi))(x\Psi) = \lambda x \lambda y x \wedge \lambda x \Phi x(x\Psi)(x\Psi) = \lambda x \lambda y y$. In other words, a model of $PP\lambda$ and a model of A.C. are never jointly embeddable in a model of the λ -calculus.

If \mathfrak{M} is a model of the typed λ -calculus then \mathfrak{M}^* solves the same functional equations as \mathfrak{M} . In particular, if \mathfrak{M} is a model of A.C. then $\mathfrak{M}^* \models A.C. \exists E$ (see Statman, 1900) so \mathfrak{M}^* is “almost” a model of A.C. If \mathfrak{M} is a model of A.C. and \mathcal{L} is a model of $PP\lambda$ then both \mathfrak{M}^* and \mathcal{L}^* are inner models of P_κ^* for $\kappa = \max\{|\mathfrak{M}_0|, |\mathcal{L}_0|\}$. Perhaps the notion of inner model will be useful in this sort of situation.

RECEIVED: February 28, 1983; ACCEPTED May 1985

REFERENCES

- ANDREWS, P. (1972), General models, descriptions, and choice in type theory, *J. Symbolic Logic* **37**, 385–394.
- BARENDREGT, H. (1981), “The Lambda Calculus,” North-Holland, Amsterdam.
- FRIEDMAN, H. (1974), Equality between functionals, Lecture Notes in Math. Vol. 453 (R. Parikh, Ed.), pp. 22–37, Springer-Verlag, Berlin/New York.
- GANDY, R. (1980), Proofs of strong normalization, in “Combinatory Logic, Lambda Calculus, and Formalism (Curry Festschrift)” (J. P. Seldin and J. R. Hindley, Eds.), Academic Press, New York.
- HIGMAN, G. (1952), Ordering by divisibility in abstract algebra, *Proc. London Math. Soc.* (3) **2**.
- HINDLEY, R. (1969), An abstract form of the Church–Rosser theorem, *J. Symbolic Logic* **34**, 545–560.
- HOWARD, W. (1973), Hereditarily majorizable functionals, Lecture Notes Vol. 344 (A. S. Troelstra, Ed.), pp. 454–461, Springer-Verlag, Berlin/New York.
- LAUCHLI, H. (1970), An abstract notion of realizability..., in “Intuitionism and Proof Theory (Buffalo Volume)” (A. Kino, J. Myhill, and R. E. Vesley, Eds.), pp. 227–234, North-Holland, Amsterdam.
- MARTIN-LÖF, P. (1973), Hauptsatz for the theory of types, in “Logic, Methodology, and the Philosophy of Science IV (Bucharest Volume)” (P. Suppes, L. Henkin, A. Joja, and C. Moisil, Eds.), pp. 279–290, North-Holland, Amsterdam.
- PLOTKIN, G. (1980), λ -definability in the full type hierarchy, in “Combinatory Logic, Lambda Calculus, and Formalism (Curry Festschrift)” (J. P. Seldin and J. R. Hindley, Eds.), pp. 363–373, Academic Press, New York.
- PLOTKIN, G. (1973), “ λ -Definability and Logical Relations,” Univ. of Edinburgh School of Artificial Intelligence Memorandum SAI-RM-4.
- PRAWITZ, D. (1971), Ideas and results in proof theory, in “Proc. 2nd Scandinavian Logic Sympos. (Oslo Volume)” (J. E. Fenstad, Ed.), pp. 235–307, North-Holland, Amsterdam.

- STATMAN, R. (1982), Completeness invariance and λ -definability, *J. Symbolic Logic* **47**, 17–26.
- STATMAN, R. (1900), Embeddings, homomorphisms and λ -definability, manuscript.
- STATMAN, R. (1900), On the existence of closed terms III, manuscript.
- STATMAN, R. (with M. Saks) (1900), Combinators, logical relations, finite automata, and Helly-type theorems manuscript.
- TAIT, W. (1967), Intentional interpretation of functionals, *J. Symbolic Logic* **32**, 198–212.